Please check out my postings, examples and extra practice on 6.4. Exam 1 will be returned Tuesday.

6.4 Work (Work = "total effort")

The concept "work" measures energy expended in completing a task. When a **constant** force is applied through a fixed distance, we define:

Work = Force · Distance (W = F · D)

Ex $\boxed{\text{I}}$ 3 lbs lifted 4 feet

$\Rightarrow$ work = 12 ft-lbs

$\boxed{\text{II}}$ 2 kg lifted 10 meters (on Earth)

Force = 2 · 9.8 = 19.6 N

$\Rightarrow$ Work = 19.6 · 10 = 196 Joules

First, some units. Newton's 2^{nd} law:

Force = Mass· Acceleration (F = m·a)

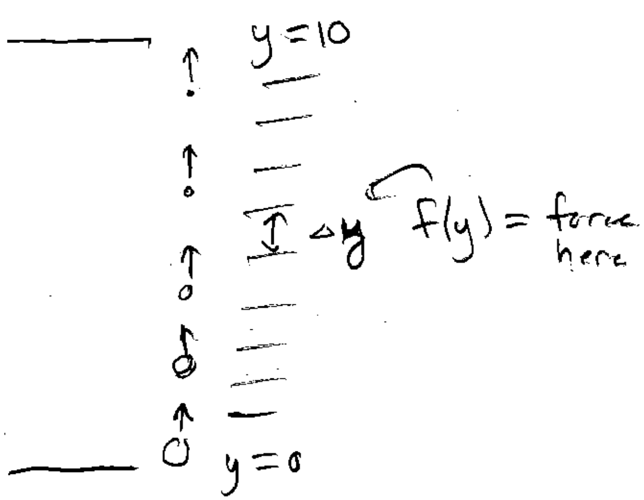
	Metric	~~**Standard**~~ IMPERIAL
Mass	kg = kilograms	
Accel. on Earth	9.8 m/s^2	32 $\frac{ft}{s^2}$
Force	N = Newtons N = kg · m/s^2	lbs = pounds
Dist	m = meters	ft = feet
Work	J = Joules J = N · m	ft-lbs

in = inch 12 in = 1 ft
yd = yard 3 ft = 1 yd
mi = miles 5280 ft = 1 mi
g = gram 1000 g = 1 kg
cm = centimeter 100 cm = 1 m

f force or distance change in some way during the task (i.e. NOT constant), then we can break up the problem into subtasks, approximate with Force · Dist on each subtask, and add up the approximations:

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} (\text{Force} \cdot \text{Distance})$$

But, we must *find the pattern* for the force and distance for each subdivision.



PROBLEM TYPE 1: *Force changing*.

Moving an object from $x = a$ to $x = b$

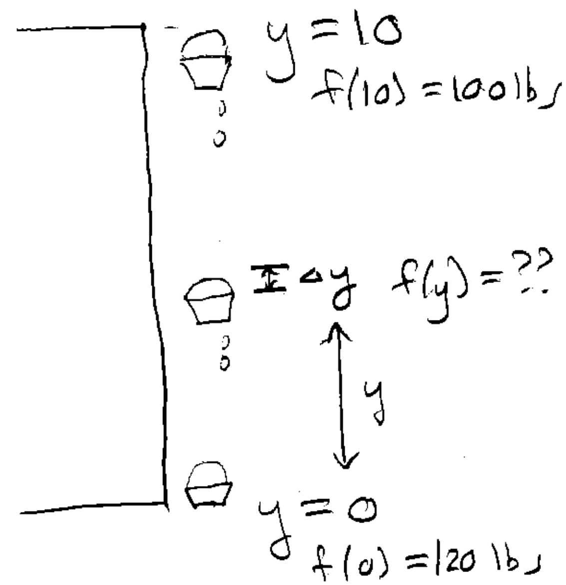
and $\quad f(x)$ = "FORCE at x"

$\quad \Delta x = \text{DISTANCE}$

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

Examples (of changing force):

. *Leaky bucket*: A leaking bucket is lifted 10 feet. At the bottom the bucket weighs 120 pounds and at the top the bucket weighs 100 pounds. Assume the water leaked out a constant rate as it was lifted.



How much work was done to lift the bucket?

constant rate ⇒ linear ⇒ $m = \frac{120-100}{0-10} = \frac{20}{-10} = -2$

$f(y) = -2(y-0) + 120$

$f(y) = -2y + 120$ lbs FORCE PATTERN

EACH WEIGHT IS LIFTED Δy, THEN IT CHANGES.

WORK $\approx f(y_1)\Delta y + f(y_2)\Delta y + f(y_3)\Delta y + \dots + f(y_{10})\Delta y$

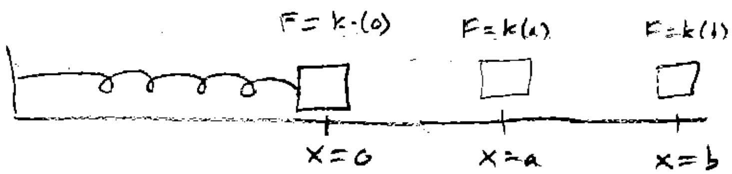
EXACT: WORK $= \int_0^{10} -2y + 120 \, dy$

$= -y^2 + 120y \Big|_0^{10} = (-(10)^2 + 120(10)) - 0 = -100 + 1200$

$= \boxed{1100 \text{ ft-lbs}}$

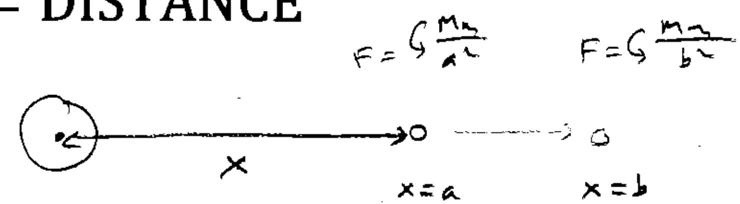
2. *Other examples where a force formula is known and the force changes every moment as the object is moved*:

- *Springs* – A weight is attached to a spring which is attached to a wall. *Hooke's law*: Force is proportional to the distance from natural length. That is, there is a constant *k* such that
  f(x) = kx ="FORCE to hold x units beyond natural length."
  $\Delta x = \text{DISTANCE}$



$$\text{Work} = \int_a^b \underbrace{kx}_{\text{FORCE}}\,\underbrace{dx}_{\text{DIST}}$$

- *Gravity* – Newton's Law of Gravitation states
  $f(x) = G\dfrac{Mm}{x^2}$
  = *"FORCE between two masses (M and m) that are x units apart"*
  (G is the gravitation constant)
  $\Delta x = \text{DISTANCE}$



$$\text{Work} = \int_a^b \underbrace{G\frac{Mm}{x^2}}_{\text{FORCE}}\,\underbrace{dx}_{\text{DIST}}$$

PROBLEM TYPE 2: *Force & dist. changing.*

n some problems, we subdivide and find

d(x) = `DISTANCE for subtask starting at *x*'

and

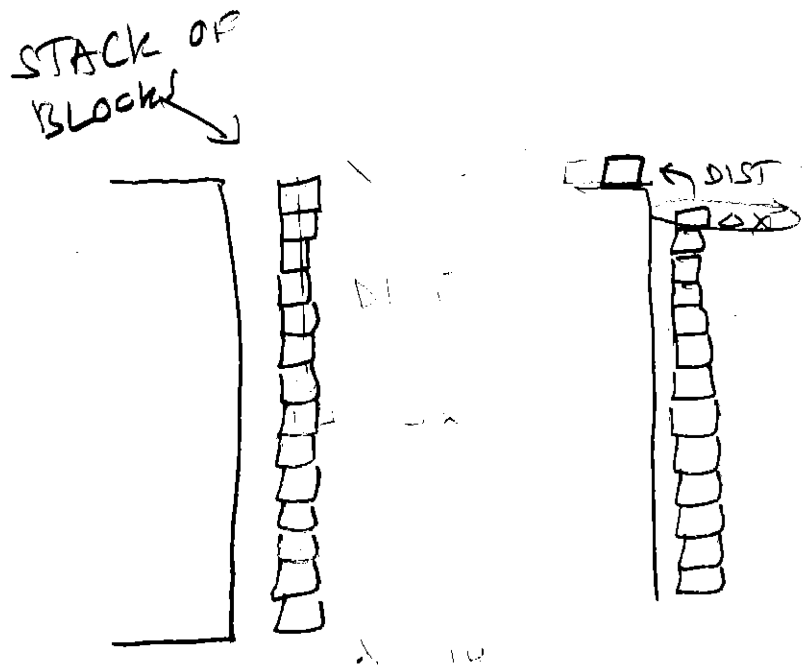
f(x) =`*density* (force/length) of subtask at *x*'

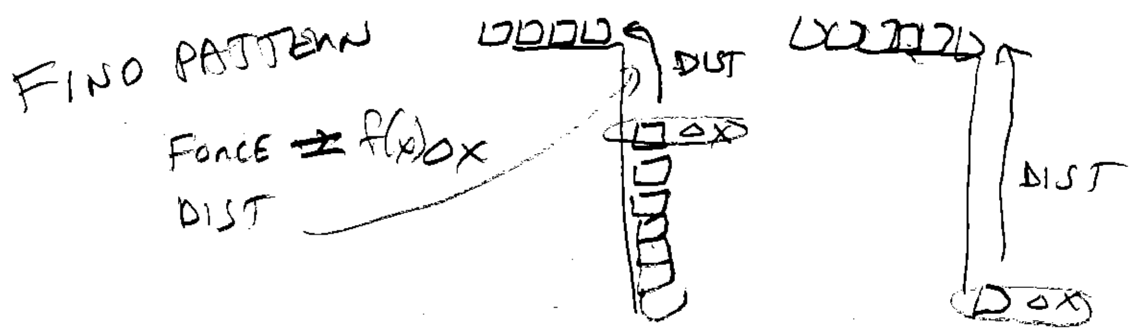
f(x)Δx = `FORCE of subtask at *x*'

n which case:

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} d(x_i) f(x_i) \Delta x$$

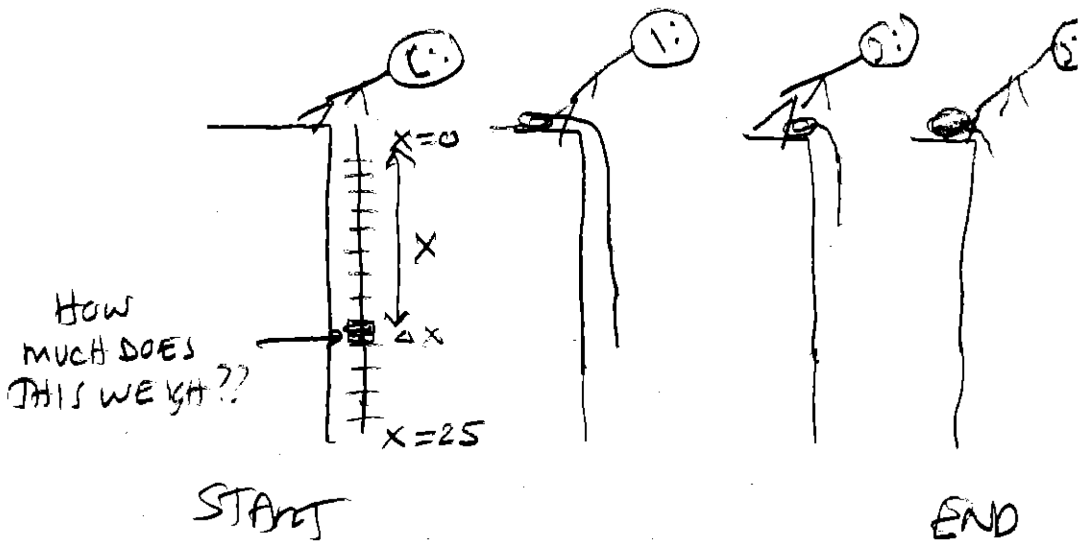
$$= \int_a^b d(x) f(x) dx$$





Examples:

1. (Chains/Cables) You are lifting a heavy chain to the top of a building. The chain has a density of 3 lbs/foot. The chain hangs over the side by 25 feet before you start pulling it up. How much work is done in pulling the chain all the way to the top?



"WEIGHT OF A HORIZONTAL SLICE" $= 3 \frac{\text{lbs}}{\text{ft}} \; \Delta x \text{ ft} = 3\Delta x \text{ lbs}$

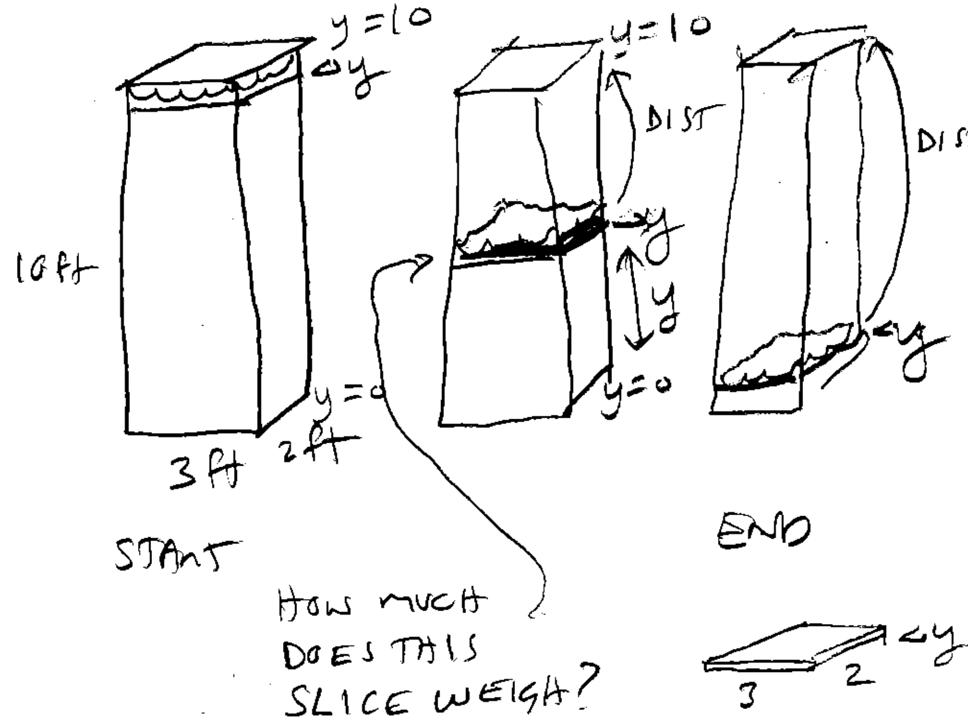
"DIST. THE A HORIZ. SLICE WILL BE LIFTED" $= x \text{ ft}$

$$\text{WORK} = \int_0^{25} x \cdot 3\,dx = \frac{3}{2}x^2 \Big|_0^{25}$$

$$= \frac{3}{2}(25)^2 = \boxed{937.5 \text{ ft-lbs}}$$

2. (Pumping Liquid) You are pumping water out of a tank. The tank is a rectangular box with a base of 2 ft by 3 ft and height of 10ft. The density of water is 62.5 lbs/ft^3.

If the tank starts full, how much work is done in pumping all the water to the top and out over the side?



"WEIGHT OF A HORIZONTAL SLICE" $= 62.5 \frac{\text{lbs}}{\text{ft}^3} \cdot (3 \cdot 2 \cdot \Delta y)\ \text{ft}^3 = 62.5 \cdot 6\, \Delta y$ lbs

"DIST THE HORIZ. SLICE IS LIFTED" $= 10 - y$ ft

$$\text{WORK} = \int_0^{10} (10-y) \cdot 62.5 \cdot 6\, dy = 62.5 \cdot 6 \int_0^{10} 10 - y\, dy$$

$$= 375 \left[10y - \tfrac{1}{2}y^2 \Big|_0^{10} \right] = 375\,[100 - 50] = \boxed{18{,}750 \text{ ft-lbs}}$$